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A Fixed-Point Approach for Nonlinear Discrete Boundary Value Problems

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Abstract—Existence results are established for second-order discrete boundary value problems.

Keywords—Discrete boundary value problems, Fixed-point methods.

1. INTRODUCTION

This paper presents existence principles and theory for second-order discrete boundary value problems where the values of the solution lie in a Banach space E (here E is not necessarily finite dimensional). The approach in this paper is based on fixed-point methods (in particular, continuation methods). Our existence argument relies on showing that no solutions of an appropriate family of problems lie on the boundary of a suitable open set. Discrete boundary value problems when $E = \mathbb{R}^m$ have been discussed widely in the literature, see [1–5] and their references. The results in this paper are in a new direction. They also extend some previously known results.

We begin in Section 2 by studying the discrete boundary value problem

$$\begin{gathered}\Delta^2 y + \mu f(i, y) = 0, \qquad i \in N,\\ \alpha_0 y(0) - \beta_0 \Delta y(0) = 0,\\ \gamma_0 y(T+1) + \delta_0 \Delta y(T+1) = 0, \qquad \alpha_0 > 0, \quad \gamma_0 > 0, \quad \beta_0 \geq 0, \quad \delta_0 \geq \gamma_0.\end{gathered} \tag{1.1}$$

Here $\mu \geq 0$, $T \in \{1, 2, \dots\}$, $N = \{0, 1, \dots, T\}$, $N^+ = \{0, 1, \dots, T+2\}$, and $y : N^+ \to \mathbb{R}^m$. We will assume throughout Section 2 that

$$f : N \times \mathbb{R}^m \to \mathbb{R}^m \text{ is continuous.} \tag{1.2}$$

REMARKS.

(i) Recall a map $f : N \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous if it is continuous as a map of the topological space $N \times \mathbb{R}^m$ into the topological space $\mathbb{R}^m$. Throughout this paper, the topology on N will be the *discrete* topology.

(ii) If $f : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous, then of course $f : N \times \mathbb{R}^m \to \mathbb{R}^m$ (i.e., f restricted to $N \times \mathbb{R}^m$) is continuous.

Let $C(N^+, \mathbb{R}^m)$ denote the class of maps w continuous on N^+ (discrete topology), with norm

$$|w|_0 = \max_{k \in N^+} |w(k)|,$$

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX i.e.,

$$C(N^+, \mathbb{R}^m) = \{w; w : N^+ \to \mathbb{R}^m\}.$$

Notice, $C(N^+, \mathbb{R}^m)$ is a Banach space.

REMARK. Since N^+ is a discrete set any mapping of N^+ to a topological space (in this case $\mathbb{R}^m$) is continuous.

By a solution to (1.1), we mean a $w \in C(N^+, \mathbb{R}^m)$ such that w satisfies (1.1) for $i \in N$ and w satisfies the boundary conditions. Section 2 develops a very general existence principle for (1.1). This principle is derived using fixed-point methods, in particular, a nonlinear alternative of Leray-Schauder type [6,7]. Some new existence theorems will then be established for (1.1) using this existence principle.

Section 3 examines (1.1) again, except in this case $y : N^+ \to E$ and

$$f : N \times E \to E \text{ is continuous.} \tag{1.3}$$

Here E is a Banach space. Again, here $C(N^+, E)$ will denote the class of maps w continuous on N^+ (discrete topology), with norm

$$|w|_0 = \max_{k \in N^+} |w(k)|; \qquad \text{here } |.| \text{ is the norm in } E.$$

We establish in Section 3 a general existence theorem for (1.1), in this case y takes values in a general Banach space. We remark here, that the results in Section 2 follow from those in Section 3. However, when E is finite dimensional the argument is a lot simpler and this is the main reason why we discuss the finite-dimensional case separately.

To conclude the Introduction, we gather together some ideas which will be used in Sections 2 and 3. We first look at the Arzela-Ascoli theorem. We could make use the well-known result in the literature [8,9]. However, in our situation the Arzela-Ascoli theorem has a somewhat simpler statement. For completeness, we state the result in $C(N^+, E)$ and we also supply a short simple proof.

THEOREM 1.1. *Let A be a closed subset of $C(N^+, E)$. If A is uniformly bounded and the set $\{u(k) : u \in A\}$ is relatively compact for each $k \in N^+$, then A is compact.*

PROOF. We need only show that every sequence in A has a Cauchy subsequence. Let $A_1 = \{f_{1,1}, f_{1,2}, \dots\}$ be any sequence in A. Notice the sequence $\{f_{1,j}(0)\}$, $j = 1, 2, \dots$ has a convergent subsequence and let $A_2 = \{f_{2,1}, f_{2,2}, \dots\}$ denote this subsequence. For $\{f_{2,j}(1)\}$, $j = 1, 2, \dots$, let $A_3 = \{f_{3,1}, f_{3,2}, \dots\}$ be the subsequence of A_2 such that $\{f_{3,j}(1)\}$ converges. Since A_3 is a subsequence of A_2, then $\{f_{3,j}(0)\}$ also converges. Continue this process to get a list of sequences

$$A_1, A_2, \dots, A_{T+3}, A_{T+4},$$

in which each sequence is a subsequence of the one directly on the left of it and for each i, the sequence $A_i = \{f_{i,1}, f_{i,2}, \dots\}$ has the property that $\{f_{i,j}(i-2)\}$, $j = 1, 2, \dots$ is a convergent sequence. Thus, for each $i \in N^+$, the sequence $\{f_{T+4,j}(i)\}$ is convergent. Then, since $\{f_{T+4,j}(i)\}$ is Cauchy for each $i \in N^+$, and since N^+ is finite, we have that there exists $n_0 \in \{1, 2, 3, \dots\}$ (independent of i) such that

$$m, n \geq n_0 \text{ implies } |f_{T+4,m}(i) - f_{T+4,n}(i)| < \epsilon, \qquad i \in N^+.$$

Thus, A_{T+4} is Cauchy and we are finished. ∎

Next, we recall some properties of measures of noncompactness [7,8,10]. Let E be a Banach space and Ω_E the bounded subsets of E. Let $X \in \Omega_E$. The diameter of X is defined by

$$\operatorname{diam}(X) = \sup\{|x - y| : x, y \in X\}; \qquad \text{here } |.| \text{ is the norm in } E.$$

The *Kuratowskii measure of noncompactness* is the map $\alpha : \Omega_E \to [0, \infty)$ defined by

$$\alpha(X) = \inf \left\{ \epsilon > 0 : X \subseteq \bigcup_{i=1}^{n} X_i \text{ and } \operatorname{diam}(X_i) \le \epsilon \right\}; \qquad \text{here } X \in \Omega_E.$$

For convenience, we recall some properties of α. Let $A, B \in \Omega_E$. Then,

(i) $\alpha(A) = 0$ iff $\bar{A}$ is compact,
(ii) $\alpha(\bar{A}) = \alpha(A)$,
(iii) if $A \subseteq B$, then $\alpha(A) \le \alpha(B)$,
(iv) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$,
(v) $\alpha(\lambda A) = |\lambda|\alpha(A)$, $\lambda \in \mathbb{R}$,
(vi) $\alpha(A + B) \le \alpha(A) + \alpha(B)$,
(vii) $\alpha(\operatorname{co}(A)) = \alpha(A)$,
(viii) for any bounded Λ of $\mathbb{R}$, we have $\alpha(\Lambda.A) = (\sup_{\lambda \in \Lambda} |\lambda|)\alpha(A)$; here $\Lambda.A = \{\lambda x : \lambda \in \Lambda, x \in A\}$.

Let E_1 and E_2 be two Banach spaces and let $F : Y \subseteq E_1 \to E_2$ be continuous and map bounded sets into bounded sets. We call F a α-Lipschitzian map if there is a constant $k \ge 0$ with $\alpha(F(X)) \le k\alpha(X)$ for all bounded sets $X \subseteq Y$. We also say F, a *Darbo* map if F is α-Lipschitzian with $k < 1$.

REMARK. If $F : Y \to E_2$ is completely continuous (i.e., the image of each bounded set in Y is contained in a compact set in E_2), then clearly F is a Darbo map (in fact, α-Lipschitzian with $k = 0$).

Finally, we state a nonlinear alternative of Leray-Schauder type for Darbo map [6,7].

THEOREM 1.2. *Let E be a Banach space with $C \subseteq E$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$, $F(\bar{U})$ bounded and $F : \bar{U} \to C$ a Darbo map. Then, either*

(A1) *F has a fixed point in $\bar{U}$; or*
(A2) *there is a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.*

2. FINITE-DIMENSIONAL CASE

In this section, we examine (1.1) with $y : N^+ \to \mathbb{R}^m$. We first establish the following general existence principle for problems of type (1.1).

THEOREM 2.1. *Suppose (1.2) is satisfied. In addition, assume there is a constant M_0 independent of λ, with*

$$|y|_0 = \sup_{i \in N^+} |y(i)| \ne M_0,$$

for any solution y (here $y \in C(N^+, \mathbb{R}^m)$) to

$$\begin{gathered} \Delta^2 y + \lambda\mu f(i, y) = 0, \qquad i \in N, \\ \alpha_0 y(0) - \beta_0 \Delta y(0) = 0, \\ \gamma_0 y(T+1) + \delta_0 \Delta y(T+1) = 0, \qquad \alpha_0 > 0, \quad \gamma_0 > 0, \quad \beta_0 \ge 0, \quad \delta_0 \ge \gamma_0, \end{gathered} \tag{2.1$_\lambda$}$$

for each $\lambda \in (0, 1)$. Then, (1.1) has a solution.

PROOF. Solving $(2.1)_\lambda$ is equivalent to finding a $y \in C(N^+, \mathbb{R}^m)$ which satisfies

$$y(i) = \lambda\mu \sum_{j=0}^{T} G(i, j) f(j, y(j)), \qquad i \in N^+, \tag{2.2$_\lambda$}$$

where

$$G(i,j) = \begin{cases} \dfrac{[\beta_0 + \alpha_0 (j+1)] [\delta_0 + \gamma_0 (T+1-i)]}{\alpha_0\gamma_0(T+1) + \alpha_0\delta_0 + \beta_0\gamma_0}, & j \in \{0,1,\dots,i-1\}, \\ \dfrac{[\beta_0 + \alpha_0 i] [\delta_0 + \gamma_0 (T-j)]}{\alpha_0\gamma_0(T+1) + \alpha_0\delta_0 + \beta_0\gamma_0}, & j \in \{i,\dots,T\}. \end{cases} \tag{2.3}$$

Define the operator $S : C(N^+, \mathbf{R}^m) \to C(N^+, \mathbf{R}^m)$ by setting

$$Sy(i) = \mu \sum_{j=0}^{T} G(i,j) f(j, y(j)).$$

Now, $(2.2)_\lambda$ is equivalent to the fixed-point problem

$$y = \lambda S y. \tag*{$(2.4)_\lambda$}$$

The continuity of f implies that $S : C(N^+, \mathbf{R}^m) \to C(N^+, \mathbf{R}^m)$ is continuous. In addition, S is completely continuous. To see this, let Ω be a bounded subset of $C(N^+, \mathbf{R}^m)$. Now, clearly $S(\Omega)$ is bounded and as a result $\overline{S(\Omega)}$ is compact (see also Theorem 1.1). Consequently, $S : C(N^+, \mathbf{R}^m) \to C(N^+, \mathbf{R}^m)$ is completely continuous. Let

$$U = \left\{u \in C\left(N^+, \mathbf{R}^m\right) : |u|_0 < M_0\right\} \quad \text{and} \quad E = C\left(N^+, \mathbf{R}^m\right).$$

Then, Theorem 1.2 applies and with the above choice of U possibility (A2) is ruled out, so S has a fixed point, i.e., (1.1) has a solution. ∎

We now use the above existence principle to establish *one* existence result for (1.1).

THEOREM 2.2. *Suppose (1.2) is satisfied. In addition assume*

there is a continuous, nondecresaing function $\psi : [0,\infty) \to [0,\infty)$, with $\psi(u) > 0$ for $u > 0$ and constants q_j, $j \in N$, with $|f(j,u)| \le q_j \psi(|u|)$, for any $u \in \mathbf{R}^m$ and $j \in N$, (2.5)

and

$$\mu_0 > 0 \textit{ satisfies } \sup_{c\in(0,\infty)} \left(\frac{c}{\mu_0 Q \psi(c)}\right) > 1; \qquad \textit{here } Q = \max_{i\in N^+} \left(\sum_{j=0}^{T} q_j G(i,j)\right), \tag{2.6}$$

hold. If $0 \le \mu \le \mu_0$, then (2.1) has a solution.

PROOF. Fix $\mu \le \mu_0$. Let $M_0 > 0$ satisfy

$$\frac{M_0}{\mu Q \psi(M_0)} > 1. \tag{2.7}$$

Let y be any solution to $(2.1)_\lambda$ for $0 < \lambda < 1$. Then, for $i \in N^+$, we have

$$\begin{aligned} |y(i)| &\le \mu \sum_{j=0}^{T} G(i,j) \left|f\left(j, y(j)\right)\right| \le \mu \sum_{j=0}^{T} q_j G(i,j) \psi\left(|y(j)|\right) \\ &\le \mu \psi\left(|y|_0\right) \sum_{j=0}^{T} q_j G(i,j) \le \mu Q \psi\left(|y|_0\right). \end{aligned}$$

Thus, $|y|_0 \le \mu Q \psi(|y|_0)$, i.e.,

$$\frac{|y|_0}{\mu Q \psi(|y|_0)} \le 1. \tag{2.8}$$

Suppose $|y|_0 = M_0$. Then, (2.8) implies

$$\frac{M_0}{\mu Q \psi(M_0)} \leq 1,$$

which contradicts (2.7). Then, any solution y of $(2.1)_\lambda$ satisfies $|y|_0 \neq M_0$. Now, Theorem 2.1 implies that (1.1) has a solution. ∎

REMARKS.

(i) The results of this section can be extended to include the discrete boundary value problem

$$\begin{gathered}\Delta^2 y + \tau y + \mu f(i,y) = 0, \qquad i \in N,\\ \alpha_0 y(0) - \beta_0 \Delta y(0) = 0,\\ \gamma_0 y(T+1) + \delta_0 \Delta y(T+1) = 0, \qquad \alpha_0 > 0, \quad \gamma_0 > 0, \quad \beta_0 \geq 0, \quad \delta_0 \geq 0,\end{gathered}$$

where τ is such that the discrete linear problem

$$\begin{gathered}\Delta^2 y + \tau y = 0, \qquad i \in N,\\ \alpha_0 y(0) - \beta_0 \Delta y(0) = 0,\\ \gamma_0 y(T+1) + \delta_0 \Delta y(T+1) = 0,\end{gathered}$$

has only the trivial solution. We remark that the only difference in the proof of Theorem 2.1 is in (2.3), i.e., the definition of the Green's function $G(i,j)$ (the existence of the Green's function for the above problem follows from [1, Chapter 9]).

(ii) Only minor adjustments are needed in the arguments in this section if we wish to discuss nonhomogeneous boundary data.

(iii) Also, only minor adjustments are needed in the arguments if we wish to discuss higher-order discrete boundary value problems.

3. GENERAL BANACH CASE

This section discusses

$$\begin{gathered}\Delta^2 y + \mu f(i,y) = 0, \qquad i \in N,\\ \alpha_0 y(0) - \beta_0 \Delta y(0) = 0,\\ \gamma_0 y(T+1) + \delta_0 \Delta y(T+1) = 0, \qquad \alpha_0 > 0, \quad \gamma_0 > 0, \quad \beta_0 \geq 0, \quad \delta_0 \geq \gamma_0,\end{gathered} \tag{3.1}$$

where y takes values in a Banach space E. Before we establish our existence result, we first prove a generalization of Theorem 1.1 (this result is in the spirit of the results in [10, Section 1.4]).

THEOREM 3.1. *Let $A \subseteq C(N^+, E)$ be bounded. Then,*

(i) $\alpha(A) = \alpha(A(N^+))$;

(ii) $\alpha(A(N^+)) = \sup_{i \in N^+} \alpha(A(i))$, *where*

$$A(i) = \{\phi(i) : \phi \in A\} \quad \text{and} \quad A(N^+) = \bigcup_{j \in N^+} \{\phi(j) : \phi \in A\}.$$

PROOF. (i) Let $\epsilon > 0$ be given. There exists a covering $\{A_i\}$, $1 \leq i \leq n$ of A such that $\operatorname{diam}(A_i) \leq \alpha(A) + \epsilon$, $i = 1, \ldots, n$. For each $j \in N^+$ let $B_{i,j} = \{\phi(j) : \phi \in A_i\}$. Now $\{B_{i,j}\}$, $1 \leq i \leq n$, $j \in N^+$ is a covering of $A(N^+)$. Fix $i \in \{1, \ldots, n\}$. For any $j \in N^+$, we have

$$\begin{aligned}\operatorname{diam}(B_{i,j}) &= \sup\{|\phi(j) - \psi(j)| : \phi, \psi \in A_i\} \leq \sup\left\{|\phi(k) - \psi(k)| : \phi, \psi \in A_i, \ k \in N^+\right\}\\ &= \operatorname{diam}(A_i) \leq \alpha(A) + \epsilon.\end{aligned}$$

Then, $\alpha(A(N^+)) \le \alpha(A) + \epsilon$ and so

$$\alpha\left(A\left(N^+\right)\right) \le \alpha(A). \tag{3.2}$$

To see the opposite inequality let $\epsilon > 0$ be given. Let $\{B_j\}$, $1 \le j \le m$ be a covering of $A(N^+)$ such that $\mathrm{diam}(B_j) \le \alpha(A(N^+)) + \epsilon$. Now, let Ω be the finite set of all maps $i \mapsto \mu(i)$ of N^+ into $\{1, 2, \dots, m\}$ and let

$$A_\mu = \{\phi \in A : \text{ for every } i \in N^+ \text{ we have } \phi(i) \in B_{\mu(i)}\}.$$

Now, $\{A_\mu\}$, $\mu \in \Omega$ is a covering of A. Let $\phi, \psi \in A_\mu, \mu$ fixed. For $i \in N^+$,

$$|\phi(i) - \psi(i)| \le \mathrm{diam}(B_{\mu(i)}) \le \alpha\left(A\left(N^+\right)\right) + \epsilon.$$

Thus, $\alpha(A) \le \alpha(A(N^+)) + \epsilon$ and so

$$\alpha(A) \le \alpha\left(A\left(N^+\right)\right). \tag{3.3}$$

The result follows from (3.2) and (3.3).

(ii) Since $A(i) \subseteq A(N^+)$ for each $i \in N^+$, then $\alpha(A(i)) \le \alpha(A(N^+))$ for each $i \in N^+$. Consequently,

$$\sup_{i \in N^+} \alpha(A(i)) \le \alpha\left(A\left(N^+\right)\right). \tag{3.4}$$

To see the opposite inequality let $\epsilon > 0$ be given. Let $\{C_j\}$, $1 \le j \le m$ be a covering of A such that $\{C_j(i)\}$, $1 \le j \le m$ is a covering of $A(i)$ satisfying

$$\max_{1 \le j \le m} [\mathrm{diam}(C_j(i))] \le \alpha(A(i)) + \epsilon.$$

Let $D_{i,j} = C_j(i)$ and $\{D_{i,j}\}$, $i \in N^+$, $1 \le j \le m$ is a covering of $A(N^+)$. Fix $i \in N^+$. For any $j \in \{1, 2, \dots, m\}$, we have

$$\begin{aligned} \mathrm{diam}(D_{i,j}) &= \sup\{|\phi(i) - \psi(i)| : \phi, \psi \in C_j\} = \mathrm{diam}(C_j(i)) \le \max_{1 \le j \le m} [\mathrm{diam}\,(C_j(i))] \\ &\le \alpha(A(i)) + \epsilon \le \sup_{k \in N^+} \alpha(A(k)) + \epsilon. \end{aligned}$$

Consequently, $\alpha(A(N^+)) \le \sup_{i \in N^+} \alpha(A(i)) + \epsilon$ and so

$$\alpha\left(A\left(N^+\right)\right) \le \sup_{i \in N^+} \alpha(A(i)). \tag{3.5}$$

The result follows from (3.4) and (3.5). ∎

THEOREM 3.2. *Suppose (1.3) is satisfied. In addition assume*

there is a continuous, nondecreasing function $\psi : [0, \infty) \to [0, \infty)$, *with* $\psi(u) > 0$ *for* $u > 0$ *and constants* q_j, $j \in N$, *with* $|f(j, u)| \le q_j \psi(|u|)$, *for any* $u \in E$ *and* $j \in N$, (3.6)

$$\mu_0 > 0 \text{ satisfies } \sup_{c \in (0,\infty)} \left(\frac{c}{\mu_0 Q \psi(c)}\right) > 1; \qquad \text{here } Q = \max_{i \in N^+} \left(\sum_{j=0}^{T} q_j G(i,j)\right), \tag{3.7}$$

$$\alpha(f(N \times A)) \le k\alpha(A) \text{ for all bounded subsets } A \text{ of } E; \text{ here } k \ge 0 \text{ is a constant}, \tag{3.8}$$

and

$$\mu_0 r(T+1)k < 1; \qquad \text{here } r = \sup_{i \in N^+} r_i \text{ and } r_i = \max_{j \in N} G(i,j), \tag{3.9}$$

hold. If $0 \le \mu \le \mu_0$, *then (3.1) has a solution.*

PROOF. Fix $\mu \le \mu_0$. Let y be any solution of

$$\begin{aligned} \Delta^2 y + \lambda\mu f(i,y) &= 0, \qquad i \in N, \\ \alpha_0 y(0) - \beta_0 \Delta y(0) &= 0, \\ \gamma_0 y(T+1) + \delta_0 \Delta y(T+1) &= 0, \end{aligned} \tag{3.10$_\lambda$}$$

for $0 < \lambda < 1$. Solving $(3.10)_\lambda$ is equivalent to finding a $y \in C(N^+, E)$ which satisfies $(2.2)_\lambda$. Define the operator $S : C(N^+, E) \to C(N^+, E)$ by setting

$$Sy(i) = \mu \sum_{j=0}^{T} G(i,j) f(j, y(j)).$$

Now, $(2.2)_\lambda$ is equivalent to the fixed-point problem $y = \lambda S y$. We *claim* $S : C(N^+, E) \to C(N^+, E)$ is a Darbo map. To see this, let Ω be a bounded subset of $C(N^+, E)$. Fix $i \in N^+$. Then, the properties of α and Theorem 3.1 yield

$$\begin{aligned} \alpha(S\Omega(i)) &\le \alpha\left(\left\{\sum_{j=0}^{T} \mu G(i,j) f(j,y(j)) : y \in \Omega\right\}\right) \\ &\le \alpha\left(\mu(T+1)\overline{\mathrm{co}}\left\{G(i,j) f\left(j, y(j)\right) : y \in \Omega,\ j \in N\right\}\right) \\ &= \mu(T+1)\alpha\left(\left\{G(i,j) f\left(j, y(j)\right) : y \in \Omega,\ j \in N\right\}\right) \\ &= \mu(T+1) r_i \alpha\left(\left\{f\left(j, y(j)\right) : y \in \Omega,\ j \in N\right\}\right) \\ &\le r_i \mu(T+1)\alpha(f(N \times \Omega(N)) \le r_i k\mu(T+1)\alpha(\Omega(N)) \\ &\le r_i k\mu(T+1)\alpha\left(\Omega\left(N^+\right)\right) = r_i k\mu(T+1)\alpha(\Omega). \end{aligned}$$

Consequently, for each $i \in N^+$

$$\alpha(S\Omega(i)) \le rk\mu(T+1)\alpha(\Omega),$$

and so, Theorem 3.1 implies

$$\alpha(S\Omega) = \sup_{i \in N^+} \alpha(S\Omega(i)) \le rk\mu(T+1)\alpha(\Omega).$$

Since $rk\mu(T+1) < 1$, then $S : C(N^+, E) \to C(N^+, E)$ is a Darbo map.

Let M_0 satisfy (2.7). Essentially, the same reasoning as in Theorem 2.2 implies

$$|y|_0 = \sup_{k \in N^+} |y(k)| \ne M_0.$$

Let

$$U = \left\{u \in C\left(N^+, E\right) : |u|_0 < M_0\right\}.$$

Apply Theorem 1.2 and we deduce immediately that S has a fixed point. ∎

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